

Static Optimization with MATLAB

Day 3: Special Cases - Optimization with Linear Constraints

GSEFM Pre-Semester Course

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Optimization with Linear Constraints I

Definition (Optimization Problem with Linear Constraints)

An optimization problem with mixed constraints consists of a set of feasible solutions M , given as:

$$M = \{x \in \mathbb{R}^n \mid Ax \leq b, A_0x = b_0\}.$$

where $g : \mathbb{R}^n \rightarrow \mathbb{R}^l$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are affine linear, given by $g(x) := Ax - b$, and $h(x) := A_0x - b_0$ with $A \in \mathbb{R}^{l \times n}$, $b \in \mathbb{R}^l$, $A_0 \in \mathbb{R}^{m \times n}$, and $b_0 \in \mathbb{R}^m$, and an objective function $f : M \rightarrow \mathbb{R}$. The goal is to find the maximum value of $f(x)$ over all $x \in M$, which is expressed as:

$$\max_{x \in M} f(x)$$

Optimization with Linear Constraints II

- If the objective function $f(x) := c^T x$ with $c \in \mathbb{R}^n$ is linear and the constraint mappings g and h are affine linear, we speak of a **linear optimization problem**.
- If the objective function $f(x) := \frac{1}{2}x^T Qx + c^T x$ with $Q \in \mathbb{R}^{n \times n}$ symmetric, $c \in \mathbb{R}^n$ is quadratic and the constraint mappings g and h are affine linear, we speak of a **quadratic optimization problem**.
- If the objective function f is nonlinear (and not quadratic) and the constraint mappings g and h are affine linear, we speak of a **linearly constrained nonlinear optimization problem**.

Karush-Kuhn-Tucker Theorem for Linear Constraints

Theorem (The Karush-Kuhn-Tucker Theorem)

Let x^ be an optimal solution to the above mentioned optimization problem with linear constraints. Then there exist Lagrange multipliers $\lambda^* \in \mathbb{R}^l$ and $\mu^* \in \mathbb{R}^m$ such that (x^*, λ^*, μ^*) is a KKT-point.*

Global Maximum if f is Concave

Theorem (Global Maximum)

If the objective function f is concave, then a point x^ is a (global=local) maximum to the above mentioned optimization problem with linear constraints if and only if there exist Lagrange multipliers $\lambda^* \in \mathbb{R}^l$ and $\mu^* \in \mathbb{R}^m$ such that (x^*, λ^*, μ^*) is a KKT-point.*

Note that $f(x) := \frac{1}{2}x^T Qx + c^T x$ is concave if Q is negative semi-definite.

References I

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